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Poissonian Obstacles with Gaussian Walls Discriminate Between Classical and Quantum Lifshits Tailing in Magnetic Fields

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We investigate the leading low-energy falloff of the integrated density of states of a charged quantum particle in the Euclidean plane subject to a perpendicular constant magnetic field and repulsive impurities randomly distributed according to Poisson's law. This so-called magnetic Lifshits tail was determined by K. Broderix *et al.* [*J. Stat. Phys.* **80**:1 (1995)] for algebraically decaying and by L. Erdős [*Probab. Theory Relat. Fields* **112**:321 (1998)] for compactly supported single-impurity potentials. While the result in the first case coincides with the corresponding classical one, the Lifshits tail in Erdős' case exhibits a genuine quantum behavior. Building on both works, we determine magnetic Lifshits tails for a wide class of positive impurity potentials with a leading long-distance decay in between these limiting cases. Gaussian decay may be shown to discriminate between classical and quantum behavior. The Lifshits tail caused by Gaussian decay reveals power-law falloff with an exponent not yet completely determined.

KEY WORDS: Random Schrödinger operators; Lifshits tails; magnetic fields.

1. INTRODUCTION

For several decades one-particle Schrödinger operators with random potentials in d-dimensional Euclidean space $\mathbb{R}^d$ ($d = 1, 2, 3,...$) have been successfully used by physicists to model quantum aspects of disordered electronic systems. The interest in low dimension ($d = 1, 2$) has been stimulated by the fabrication of semiconductor microstructures and microdevices as well as by the discovery of the (integer) quantum Hall effect. In this context, both theoretical physicists and—more recently—

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mathematicians have intensively analyzed random Schrödinger operators for the plane $\mathbb{R}^2$ and with a perpendicular constant magnetic field. These operators act on the Hilbert space $L^2(\mathbb{R}^2)$ of Lebesgue square-integrable, complex-valued functions on the plane and are informally given by differential expressions of the form

$$H(V_\omega) := H(0) + V_\omega \tag{1.1}$$

where the unperturbed part $H(0)$ is non-random and given by the usual Landau Hamiltonian (in the symmetric gauge)

$$H(0) := \varepsilon_0 \left[\left(\mathrm{i}\ell \frac{\partial}{\partial x_1} - \frac{x_2}{2\ell} \right)^2 + \left(\mathrm{i}\ell \frac{\partial}{\partial x_2} + \frac{x_1}{2\ell} \right)^2 \right] \tag{1.2}$$

Here i stands for the imaginary unit and (x_1, x_2) for the pair of Cartesian co-ordinates of a given point $x \in \mathbb{R}^2$ interpreted as the classical position of a particle with mass $m > 0$ and electric charge $Q \neq 0$. Furthermore, $2m\varepsilon_0/|Q|\,\hbar > 0$ is the strength of the magnetic field and $\ell := \hbar/\sqrt{2m\varepsilon_0}$ stands for the so-called magnetic length where $2\pi\hbar > 0$ denotes Planck's constant.

The explicit spectral resolution

$$H(0) = \sum_{n=0}^{\infty} \varepsilon_n P_n \tag{1.3}$$

of the unperturbed operator dates back to Fock[7] and Landau,[11] where the eigenvalue $\varepsilon_n := (2n+1)\,\varepsilon_0$ is called the nth Landau level and the associated infinite-dimensional orthogonal eigenprojection P_n is an integral operator given by the kernel

$$P_n(x, y) := \frac{1}{2\pi\ell^2} \exp\left[\mathrm{i}\,\frac{x_2 y_1 - x_1 y_2}{2\ell^2} - \frac{|x-y|^2}{4\ell^2} \right] \mathrm{L}_n\left(\frac{|x-y|^2}{2\ell^2} \right) \tag{1.4}$$

Here $|x-y|^2 := (x_1 - y_1)^2 + (x_2 - y_2)^2$ denotes the square of the Euclidean distance between the points $x, y \in \mathbb{R}^2$ and $\xi \mapsto \mathrm{L}_n(\xi) := (1/n!)\, \mathrm{e}^{\xi} (\mathrm{d}^n/\mathrm{d}\xi^n) (\xi^n\, \mathrm{e}^{-\xi})$ is the nth Laguerre polynomial.

Throughout this paper the random potential V_ω perturbing $H(0)$ in (1.1) is chosen to be a repulsive Poissonian one. It acts as a multiplication operator and can be defined informally as

$$V_\omega(x) := \sum_j U(x - q_\omega(j)), \qquad U \geqslant 0 \tag{1.5}$$

Here for a given realization $\omega \in \Omega$ of the randomness the point $q_\omega(j) \in \mathbb{R}^2$ stands for the position of the jth impurity repelling the particle at $x \in \mathbb{R}^2$

by a positive potential U which neither depends on ω nor on j. We assume that $U>0$ on some open set in $\mathbb{R}^2$. The randomness is chosen such that the probability of simultaneously finding $M_1, M_2,..., M_K$ impurity points in respective pairwise disjoint subsets $\Lambda_1, \Lambda_2,..., \Lambda_K \subset \mathbb{R}^2$ is given by the product $\prod_{k=1}^{K} \mathrm{e}^{-\varrho\,|\Lambda_k|}(\varrho\,|\Lambda_k|)^{M_k}/M_k!$, where $|\Lambda_k|$ is the area of Λ_k and the parameter $\varrho>0$ is the mean concentration of impurities. From the particle's classical point of view and in the terminology of ref. 17 the impurities appear as soft Poissonian obstacles with shape function U. More or less fast decay of U at infinity is "felt" as a more or less steep wall bordering an individual obstacle.

The spectral properties of the perturbed Schrödinger operator (1.1) are still insufficiently understood. While it is known that its spectrum is with probability one non-random and equal to the half-line $[\varepsilon_0, \infty[$, many details such as spectral localization, that is, the existence of a pure-point component (away from the Landau levels) are still not settled. The simplest but physically important spectral characteristics of the given Schrödinger operator is its integrated density of states N: $E \mapsto N(E)$. Roughly speaking, $N(E)$ is the averaged number of energy levels per area below a given energy $E \in \mathbb{R}$. In particular, the set of growth points of N coincides with the almost-sure spectrum $[\varepsilon_0, \infty[$.

In this paper we study N near the bottom of the almost-sure spectrum, that is, the behavior of $N(\varepsilon_0 + E)$ for $E \searrow 0$. Since the presence of the obstacles rarefies low-lying energy levels, one expects that the values of N near ε_0 are dramatically diminished. The resulting leading low-energy falloff is commonly referred to as a Lifshits tail. In the first place, it depends on the long-distance decay of U. In fact, the slower the decay of U, the faster the falloff of N. In a wider sense, the notion Lifshits tail is also used for the low-energy behavior of the integrated density of states in case of non-positive impurity potentials.

For vanishing magnetic field the Lifshits tails have been identified for various types of impurity potentials—first by I. M. Lifshits (1917–1982) himself and, most recently, in ref. 9. We refer to this work, the monographs refs. 4, 14 and refs. 8, 2 for long lists of references and part of the history of the subject.

For non-vanishing magnetic field the Lifshits tails have been rigorously determined, up to now, only for positive U with extremely slow or extremely fast long-distance decay in refs. 2 and 6, respectively. More precisely, in case of a definite algebraic (but integrable) long-distance decay, that is

$$\lim_{|x| \to \infty} |x|^\alpha\, U(x) = u \tag{1.6}$$

for some constants $0 < u < \infty$ and $\alpha > 2$, the integrated density of states was shown[2] to fall off to zero at ε_0 faster than any power of E in the sense that[2]

$$\lim_{E \searrow 0} E^{2/(\alpha-2)} \ln N(\varepsilon_0 + E) = -C(\alpha, u, \varrho) \tag{1.7}$$

In this case, the leading asymptotics of N at the bottom ε_0 of the almost-sure spectrum coincides with that of the corresponding classical integrated density of states N_c at the infimum of its support, that is, at zero energy. In this sense, we refer to (1.7) as an example of a classical Lifshits tail. An indication of the classical behavior of (1.7) is provided by the fact that the explicitly known $C > 0$ depends only on the decay exponent α, the asymptotic constant u and the mean concentration ϱ of the impurities but not on the magnetic field or Planck's constant, see (2.24) in ref. 2. Roughly speaking, $N_c(E)$ is the average volume per area of that part in phase space $\mathbb{R}^2 \times \mathbb{R}^2$ on which the classical Hamiltonian

$$h_\omega(x, p) := \frac{1}{2m}\left[\left(p_1 + \frac{\hbar}{2\ell^2} x_2\right)^2 + \left(p_2 - \frac{\hbar}{2\ell^2} x_1\right)^2\right] + V_\omega(x) \tag{1.8}$$

corresponding to (1.1) takes on values smaller than the given energy E.

Reference 6 is concerned with a more challenging quantum Lifshits tail. In developing a version of the "method of enlargement of obstacles"[17] for obtaining the long-time asymptotics of Brownian motion among Poissonian obstacles, Erdős[6] proved that for positive (and continuous) U with compact support, in other words with finite range, one has power-law falloff at ε_0 in the sense that

$$\lim_{E \searrow 0} \frac{\ln N(\varepsilon_0 + E)}{|\ln E|} = -2\pi\varrho\ell^2 \tag{1.9}$$

Remarkably, the exponent $2\pi\varrho\ell^2$ is just the mean number of inpurities in a disk of radius $\sqrt{2}\,\ell$.

Building on the result (1.9) and techniques of ref. 2, our goal in the present paper is to derive the magnetic Lifshits tails for a wide class of positive impurity potentials U with long-distance decay in between these two extreme cases. It will turn out that all impurity potentials U which decay faster than any Gaussian lead to the same quantum Lifshits tail given by (1.9). On the other hand, we identify explicitly classical Lifshits tails for all U with so-called stretched-Gaussian decay. More generally, the techniques of ref. 2 and this paper show that the Lifshits tails coincide with the corresponding classical ones in basically all cases in which U decays

[2] Here and in the following we suppress suitable constants ensuring dimensionless arguments of the natural logarithm because they become irrelevant in the limit.

slower than any Gaussian. As a consequence, the decay of U that discriminates between quantum and classical Lifshits tailing in magnetic fields is seen to be Gaussian. This has already been conjectured by Erdős.[6] Although we offer a conjecture on the value of the exponent for the power-law Lifshits tail caused by Gaussian decay, its definite value remains open.

The present paper is organized as follows. The next section contains precise definitions, assumptions, the main theorems and related results. The proofs are given in Section 3. For completeness, the Appendix presents a Tauberian theorem needed in Section 3.

2. MAIN RESULTS

2.1. Assumptions and Definitions

We will consider throughout positive impurity potentials $U: \mathbb{R}^2 \to [0, \infty[$, which are locally square integrable,

$$U \geqslant 0, \qquad U \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2) \tag{2.1}$$

and strictly positive on a non-empty open set, that is

$$U(x) \geqslant g_0 \Theta(r - |x-a|) := g_0 \begin{cases} 1 & \text{if} \quad |x-a| < r \\ 0 & \text{if} \quad |x-a| \geqslant r \end{cases} \tag{2.2}$$

for some g_0, $r > 0$ and $a \in \mathbb{R}^2$. Moreover U is assumed to possess one of the following three decay properties at infinity:

Stretched-Gaussian decay. There exists $0 < \lambda < \infty$ and $0 < \alpha < 2$ such that

$$\lim_{|x| \to \infty} \frac{\ln U(x)}{|x|^\alpha} = -\frac{1}{\lambda^\alpha} \tag{2.3}$$

Super-Gaussian decay.

$$\lim_{|x| \to \infty} \frac{\ln U(x)}{|x|^2} = -\infty \tag{2.4}$$

Gaussian decay. There exists $0 < \lambda < \infty$ such that

$$\lim_{|x| \to \infty} \frac{\ln U(x)}{|x|^2} = -\frac{1}{\lambda^2} \tag{2.5}$$

Remark 2.1. (i) The class of functions with super-Gaussian decay defined by (2.4) consists of all functions which decay faster at infinity than any Gaussian, that is, than a Gaussian with an arbitrary decay length. In particular, with the convention $\ln 0 := -\infty$, the set of compactly supported functions is included. In contrast, while all functions in the class with stretched-Gaussian decay (2.3) fall off to zero slower than any Gaussian, this class covers only functions which are of the form $U(x) = g\exp\{-(|x|/\lambda)^{\alpha}(1+o(1))\}$ for $|x| \to \infty$, where $g > 0$, $\alpha < 2$ and "little oh" $o(1)$ stands for any function decaying to zero. Of course, for $\alpha = 1$ stretched-Gaussian decay covers exponential decay which is of physical relevance in the context of screening of charged impurities.

(ii) The class of functions with Gaussian or super-Gaussian decay is naturally complemented by the class of functions U with *sub-Gaussian decay*, that is

$$\lim_{|x|\to\infty} \frac{|x|^2}{\ln U(x)} = -\infty \tag{2.6}$$

It consists of all functions which decay slower than any Gaussian. For instance, the functions with algebraic decay (1.6) or stretched-Gaussian decay (2.3) are contained in this class. An example of a function with sub-Gaussian decay which has neither algebraic nor stretched-Gaussian decay is $U(x) = g\exp[-(|x|/\lambda)\ln(|x|/\mu)]$, $g, \lambda, \mu > 0$.

(iii) Gaussian (2.5), stretched-Gaussian (2.3) and, more generally, sub-Gaussian decay (2.6) imply strict positivity of U on some open set as is required in (2.2).

(iv) Either of the decay properties (2.3)–(2.5) together with (2.1) guarantees that U is both integrable and square integrable

$$U \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{L}^2(\mathbb{R}^2) \tag{2.7}$$

In combination with $U \geqslant 0$ this implies that the Poissonian potential (1.5) can be rigorously defined as a positive, measurable, ergodic random field on $\mathbb{R}^2$ with an underlying complete probability space $(\Omega, \mathscr{A}, \mathbb{P})$. Moreover, the Laplace characteristic functional

$$\begin{aligned} &\mathbb{E}\left[\exp\left(-\int_{\mathbb{R}^2} J(\mathrm{d}^2x)\, V_\omega(x)\right)\right] \\ &\quad = \exp\left\{-\varrho\int_{\mathbb{R}^2}\mathrm{d}^2x\left[1-\exp\left(-\int_{\mathbb{R}^2} J(\mathrm{d}^2y)\, U(x-y)\right)\right]\right\} \end{aligned} \tag{2.8}$$

exists for all finite positive Borel measures J on $\mathbb{R}^2$; confer ref. 14, Eq. (1.40) and Proposition 1.16. Here $\mathbb{E}(\cdot) := \int_\Omega \mathrm{d}\mathbb{P}(\omega)(\cdot)$ denotes the expectation with respect to the probability measure $\mathbb{P}$.

(v) By employing the $\mathrm{L}^p(\mathbb{R}^2)$-norm of the impurity potential

$$\| U \|_p := \left(\int_{\mathbb{R}^2} \mathrm{d}^2 x \, |U(x)|^p \right)^{1/p}, \qquad p \in [1, 2], \tag{2.9}$$

the expectation value

$$\mathbb{E}\left[\int_\Lambda \mathrm{d}^2 x \, (V_\omega(x))^2 \right] = \varrho \, |\Lambda| \, (\| U \|_2^2 + \varrho \, \| U \|_1^2) \tag{2.10}$$

is seen to be finite for any compact set $\Lambda \subset \mathbb{R}^2$ of Lebesgue measure $|\Lambda| := \int_\Lambda \mathrm{d}^2 x$, which implies that $V_\omega \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$ for $\mathbb{P}$-almost all $\omega \in \Omega$. With the help of Theorem 1.15 in ref. 5 one thus shows that the Schrödinger operator $H(V_\omega)$ is for $\mathbb{P}$-almost all $\omega \in \Omega$ essentially self-adjoint on $\mathscr{C}_0^\infty(\mathbb{R}^2) \subset \mathrm{L}^2(\mathbb{R}^2)$, the dense subspace of complex-valued, compactly supported, arbitrarily often differentiable functions on $\mathbb{R}^2$. Moreover, $H(V_\omega)$ is ergodic and Proposition V.3.1 of ref. 4 ensures that $\omega \mapsto H(V_\omega)$ is measurable.

The object of interest in this paper, the integrated density of states N, may be defined by the expectation value

$$N(E) := \mathbb{E}[\Theta(E - H(V_\omega))(x, x)], \qquad E \in \mathbb{R} \tag{2.11}$$

where $\mathbb{R}^2 \times \mathbb{R}^2 \ni (x, y) \mapsto \Theta(E - H(V_\omega))(x, y) \in \mathbb{C}$, denotes the complex-valued continuous integral kernel of the spectral projection $\Theta(E - H(V_\omega))$. In fact, N is the distribution function of a positive Borel measure on the real line $\mathbb{R}$ with topological support equal to $[\varepsilon_0, \infty[$, the spectrum of $H(V_\omega)$ for $\mathbb{P}$-almost all ω.

Remark 2.2. (i) Theorem 6.1 and Remark 6.2(ii) of ref. 3 show that spectral projections of $H(V_\omega)$ indeed possess continuous integral kernels for $\mathbb{P}$-almost all ω, see also Lemma 3.1 in ref. 18. Due to (magnetic) translation invariance the right-hand side of (2.11) is independent of the chosen $x \in \mathbb{R}^2$ such that

$$N(E) = \frac{1}{|\Lambda|} \, \mathbb{E}\left[\int_\Lambda \mathrm{d}^2 x \, \Theta(E - H(V_\omega))(x, x) \right] \tag{2.12}$$

for any bounded open square $\Lambda \subset \mathbb{R}^2$.

(ii) Definition (2.11) of the integrated density of states coincides with the more physical one by means of a spatial average in the macroscopic limit. More precisely, by the ergodicity of V_ω one has the identity

$$N(E) = \lim_{\Lambda \nearrow \mathbb{R}^2} \frac{1}{|\Lambda|} \int_\Lambda \mathrm{d}^2x \, \Theta(E - H_\Lambda(V_\omega))(x, x) \tag{2.13}$$

for $\mathbb{P}$-almost all $\omega \in \Omega$ and for all continuity points E of N. Here the finite-area operator $H_\Lambda(V_\omega)$ is the restriction of the Schrödinger operator (1.1) to a bounded open square $\Lambda \subset \mathbb{R}^2$ with zero Dirichlet boundary conditions. It can be rigorously defined on $\mathrm{L}^2(\Lambda)$ as a sum of quadratic forms. Its spectrum is purely discrete $\mathbb{P}$-almost surely such that the pre-limit expression in (2.13), which is equal to $|\Lambda|^{-1}$ times the number of eigenvalues of $H_\Lambda(V_\omega)$ smaller than E, is finite. The fact that the right-hand side of (2.13) is non-random $\mathbb{P}$-almost surely goes often under the name self-averaging. For details see Theorem 2.2 in ref. 12a, Proposition 3.2 in ref. 18, Remark 2.2(ii) in ref. 2, ref. 6 and references therein.

(iii) Replacing in (2.12) the quantum Hamiltonian (1.1) by its classical counterpart (1.8) and the trace over Hilbert space by an integration over phase space, we define the (quasi-)classical integrated density of states by

$$N_c(E) := \frac{1}{|\Lambda|} \mathbb{E}\left[\int_{\Lambda \times \mathbb{R}^2} \frac{\mathrm{d}^2x \, \mathrm{d}^2p}{(2\pi\hbar)^2} \, \Theta(E - h_\omega(x, p))\right] \tag{2.14}$$

In accordance with a theorem[13] of N. Bohr and J. H. van Leeuwen on the non-existence of diamagnetism in classical physics, the integration with respect to the canonical momentum $p \in \mathbb{R}^2$ shows that $N_c(E)$ does not depend on the magnetic-field strength $2m\varepsilon_0/|Q|\,\hbar = \hbar/|Q|\,\ell^2$. Translation invariance then simply gives

$$N_c(E) = \frac{m}{2\pi\hbar^2} \mathbb{E}[(E - V_\omega(0)) \, \Theta(E - V_\omega(0))] \tag{2.15}$$

independent of the chosen $\Lambda \subset \mathbb{R}^2$.

2.2. The Magnetic Lifshits Tails

Having stated the assumptions and the main definitions we are prepared to formulate the following three theorems which contain our main results, namely the magnetic Lifshits tails for stretched-Gaussian, super-Gaussian and Gaussian decay of the impurity potential. Unfortunately, for Gaussian decay our result is not quite complete.

Theorem 2.3. For a positive impurity potential with stretched-Gaussian decay (2.3) satisfying (2.1) the leading low-energy asymptotics of the integrated density of states reads

$$\lim_{E \searrow 0} \frac{\ln N(\varepsilon_0 + E)}{|\ln E|^{2/\alpha}} = -\pi \varrho \lambda^2 \tag{2.16}$$

Theorem 2.4. For a positive impurity potential with super-Gaussian decay (2.4) satisfying (2.1) and (2.2) the leading low-energy asymptotics of the integrated density of states reads

$$\lim_{E \searrow 0} \frac{\ln N(\varepsilon_0 + E)}{|\ln E|} = -2\pi \varrho \ell^2 \tag{2.17}$$

Theorem 2.5. For a positive impurity potential with Gaussian decay (2.5) satisfying (2.1) the leading low-energy asymptotics of the integrated density of states is bounded according to the inequalities

$$\begin{aligned} -\pi\varrho(\lambda^2 + 2\ell^2) &\leqslant \liminf_{E \searrow 0} \frac{\ln N(\varepsilon_0 + E)}{|\ln E|} \\ &\leqslant \limsup_{E \searrow 0} \frac{\ln N(\varepsilon_0 + E)}{|\ln E|} \leqslant -\pi\varrho \max\{\lambda^2, 2\ell^2\} \end{aligned} \tag{2.18}$$

Remark 2.6. (i) For stretched-Gaussian decay of the impurity potential the Lifshits tail reveals a classical behavior, similarly to the case with algebraic decay. More precisely, as is the case with (1.7), Eq. (2.16) remains valid if one substitutes $N_c(E)$, given by (2.15), for $N(\varepsilon_0 + E)$. In particular, the right-hand side of (2.16) is independent of the magnetic field although the latter must be non-zero. More generally, classical Lifshits tailing may be shown to occur basically for all impurity potentials in $L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ with sub-Gaussian decay (2.6). The argument for this is as follows. Inspecting the subsequent proof of Theorem 2.3 one sees that classical Lifshits tailing occurs if the decay of U survives the convolution with a Gaussian. But this is the case if U has a definite sub-Gaussian decay without severe oscillations.[3] One should notice that the details of a classical Lifshits tail depend on the specific (sub-Gaussian) decay of U, confer, for example, (1.7) and (2.16).

(ii) All impurity potentials with a super-Gaussian decay cause the same Lifshits tail (2.17) with a genuine quantum behavior, namely that

[3] See Note Added in Proof.

first found by Erdős[6] for compactly supported impurity potentials, confer (1.9).

(iii) As a consequence of (i) and (ii), Gaussian decay turns out to be the discriminating decay between classical and quantum Lifshits tailing in magnetic fields. By comparison, we recall (see Corollary 9.14 and Theorem 10.2 in ref. 14) that for vanishing magnetic field the discriminating decay is (for two spatial dimensions) inverse quartic, that is, algebraic with $\alpha = 4$, confer (1.6).

(iv) It is a widespread belief in theoretical physics (see for example ref. 12, Eq. (17.3) and ref. 14, Eq. (9.44a)) that the behavior of N at the bottom η of the almost-sure spectrum of $H(V_\omega)$ is universally given for $E \searrow 0$ by the formula

$$\ln N(\eta + E) = \Big\{ \inf_{t>0} \Big[t(\eta + E) + \sup_{\substack{\psi \in \mathrm{L}^2(\mathbb{R}^d) \\ \langle \psi, \psi \rangle = 1}} \ln \mathbb{E}(\mathrm{e}^{-t\langle \psi, H(V_\omega)\psi \rangle}) \Big] \Big\} (1 + o(1)) \tag{2.19}$$

where $\langle \varphi, \psi \rangle := \int_{\mathbb{R}^d} \mathrm{d}^d x \, \varphi^*(x) \, \psi(x)$ denotes the standard inner product of $\varphi, \psi \in \mathrm{L}^2(\mathbb{R}^d)$. The subsequent proof shows that the asymptotics presented in Theorems 2.3 and 2.4 as well as the lower bound in Theorem 2.5 are consistent with this formula. Therefore we conjecture that the lower bound in (2.18) reflects the true asymptotics of N for Gaussian decay. Obviously, it becomes purely classical only in the limit $\ell \searrow 0$.

(v) For all three types of decay considered in Theorems 2.3–2.5, the integrated density of states N is continuous at the bottom of the spectrum. Using Lemma V.2.1 and Lemma V.2.12 of ref. 4 (see also Section 4.3 in ref. 8) one thus proves that the infinite degeneracy of the lowest Landau level is completely lifted $\mathbb{P}$-almost surely. In fact, ε_0 is not even an eigenvalue of $H(V_\omega)$ for $\mathbb{P}$-almost all ω. Note that this statement holds irrespectively of how large the support of the impurity potential is if only it contains a non-empty open set. Over against this, in case of Poissonian *point* impurities $U(x) = u\delta(x)$, $u > 0$, and when $H(V_\omega)$ is restricted to the eigenspace $P_0 \mathrm{L}^2(\mathbb{R}^2)$ of the lowest Landau level, it was shown in ref. 6 that $\mathbb{P}$-almost surely ε_0 remains an infinitely degenerate eigenvalue provided that $2\pi\varrho\ell^2 < 1$. In this context, see also refs. 1, 5a, 15, and 16.

2.3. Note on the Restricted Integrated Density of States

Instead of N one often investigates the so-called restricted integrated density of states

$$R_n(E) := \mathbb{E}[(P_n \Theta(E - P_n H(V_\omega) P_n) P_n)(x, x)] \tag{2.20}$$

of the Schrödinger operator (1.1) restricted to the eigenspace $P_n \mathrm{L}^2(\mathbb{R}^2)$ of the nth Landau level, see for example refs. 1, 2, 6, 10, 15, 16 and references therein. According to ref. 2 the restricted operator $P_n H(V_\omega) P_n = \varepsilon_n P_n + P_n V_\omega P_n$ is self-adjoint for $\mathbb{P}$-almost all ω if U is locally bounded which is a stronger condition than $U \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$, confer (2.1).

Concerning the leading low-energy asymptotics of R_n, we only have the following results:

$$\lim_{E \searrow 0} \frac{\ln R_n(\varepsilon_n + E)}{|\ln E|^{2/\alpha}} = -\pi \varrho \lambda^2 \tag{2.21}$$

for stretched-Gaussian,

$$-2\pi \varrho \ell^2 \leqslant \liminf_{E \searrow 0} \frac{\ln R_n(\varepsilon_n + E)}{|\ln E|} \tag{2.22}$$

for super-Gaussian and

$$\begin{aligned} -\pi \varrho(\lambda^2 + 2\ell^2) &\leqslant \liminf_{E \searrow 0} \frac{\ln R_n(\varepsilon_n + E)}{|\ln E|} \\ &\leqslant \limsup_{E \searrow 0} \frac{\ln R_n(\varepsilon_n + E)}{|\ln E|} \leqslant -\pi \varrho \lambda^2 \end{aligned} \tag{2.23}$$

for Gaussian decay of the impurity potential U.

We omit the proofs of (2.21)–(2.23), because they follow mostly the strategy of the proofs of Theorems 2.3–2.5 given in the next section. In fact, the proofs of (2.21)–(2.23) are based on the inequalities (3.7) and (3.8) of ref. 2, which take the place of the subsequent inequalities (3.5) and (3.4), respectively. The reason why we do not have stringent upper bounds for super-Gaussian and Gaussian decay is that in the restricted case we are not aware of a result taking the place of Corollary 3.3.

For similar reasons as in the unrestricted case, we conjecture that the left-hand sides in (2.22) and (2.23) reflect the true asymptotics of R_n. In particular, in case of (2.22) with $n = 0$ this conjecture is supported by arguments given implicitly in ref. 6.

Remark 2.7. In ref. 10 the restricted integrated density of states R_0 for a Gaussian impurity potential $U(x) = (u/\pi\lambda^2) \exp(-|x|^2/\lambda^2)$, $u, \lambda > 0$, is approximately constructed from the first twelve moments of the probability distribution function $2\pi\ell^2 R_0$. Contrary to one of our conjectures, its authors suggest that the leading low-energy falloff should be given by

$\lim_{E \searrow 0} |\ln E|^{-1} \ln R_0(\varepsilon_0 + E) = 1 - \pi\varrho(\lambda^2 + 2\ell^2)$ if $2\pi\varrho\ell^2 \geqslant 1$. This suggestion is consistent with (2.23) and reproduces (for $2\pi\varrho\ell^2 \geqslant 1$) the known result(1) for Poissonian point impurities in the limit $\lambda \searrow 0$.

3. PROOFS

The proofs of Theorems 2.3–2.5 rely on upper and lower bounds on the low-energy asymptotics of the integrated density of states, the basic inequalities for which will be given first.

3.1. Basic Inequalities

For the proof of classical Lifshits tailing, as claimed (for example) under the assumptions of Theorem 2.3, we follow the strategy of ref. 2. Instead of N we investigate its shifted Laplace transform

$$\widetilde{N}(t) := \int_0^\infty \mathrm{d}N(\varepsilon_0 + E)\, \mathrm{e}^{-tE}, \qquad t > 0 \tag{3.1}$$

in the long-time limit $t \to \infty$ and use a Tauberian theorem which will be proved in the Appendix.

As expected for classical behavior, a simple Golden–Thompson type of upper bound on $\widetilde{N}$ already reflects its leading long-time falloff. In this context it is useful to note that

$$\frac{m}{2\pi\hbar^2 t}\, \mathbb{E}[\mathrm{e}^{-tV_\omega(0)}] = \int_0^\infty \mathrm{d}N_c(E)\, \mathrm{e}^{-tE}, \qquad t > 0 \tag{3.2}$$

is the Laplace transform of the classical integrated density of states (2.15). A lower bound on $\widetilde{N}$ is provided by a Berezin–Lieb–Luttinger type of inequality which is sharper than inequality (3.5) in ref. 2 if and only if $t > (2\varepsilon_0)^{-1}$. Its formulation and proof makes use of the two-parameter family of complex-valued, normalized Gaussian functions

$$y \mapsto \phi_x(y) := \sqrt{2\pi\ell^2}\, P_0(y, x), \qquad x \in \mathbb{R}^2, \qquad \langle \phi_x, \phi_x \rangle = 1 \tag{3.3}$$

which belong to the ground-state eigenspace $P_0\mathrm{L}^2(\mathbb{R}^2)$ of the Landau Hamiltonian $H(0)$, confer (1.4). Both basic inequalities are summarized in the following

Theorem 3.1. The shifted Laplace transform $\tilde{N}$ is bounded pointwise according to

$$\tilde{N}(t) \leqslant \frac{e^{t\varepsilon_0}}{4\pi\ell^2 \sinh(t\varepsilon_0)} \mathbb{E}[e^{-tV_\omega(0)}] \tag{3.4}$$

$$\frac{1}{2\pi\ell^2} \mathbb{E}[e^{-t\langle\phi_0, V_\omega\phi_0\rangle}] \leqslant \tilde{N}(t) \tag{3.5}$$

Proof. The Golden–Thompson type of inequality (3.4) was proved in ref. 2. For the proof of (3.5) we start by adding a quadratic confining potential thereby introducing the total potential

$$V_{\omega, D}(x) := V_\omega(x) + D\,|x|^2 \tag{3.6}$$

with some $D > 0$. Then for $t > 0$ the "Boltzmann operator" $\exp[-tH(V_{\omega, D})]$ is trace class for $\mathbb{P}$-almost all $\omega \in \Omega$, and one finds for its trace

$$\operatorname{Tr} e^{-tH(V_{\omega, D})} = \sum_{n=0}^{\infty} \operatorname{Tr} P_n\, e^{-tH(V_{\omega, D})} P_n \geqslant \operatorname{Tr} P_0\, e^{-tH(V_{\omega, D})} P_0 \tag{3.7}$$

by omitting several positive terms. Since $P_0 \exp\{-tH(V_{\omega, D})\} P_0$ possesses a continuous integral kernel given by $(x, y) \mapsto \langle\phi_x, \exp\{-tH(V_{\omega, D})\}\, \phi_y\rangle / 2\pi\ell^2$, we may write

$$\operatorname{Tr} P_0\, e^{-tH(V_{\omega, D})} P_0 = \frac{1}{2\pi\ell^2} \int_{\mathbb{R}^2} d^2x\, \langle\phi_x, e^{-tH(V_{\omega, D})}\, \phi_x\rangle \tag{3.8}$$

With the help of the Jensen–Peierls inequality the integrand can be bounded from below as follows

$$\begin{aligned} &\langle\phi_x, e^{-tH(V_{\omega, D})}\, \phi_x\rangle \\ &\quad \geqslant \exp[-t\langle\phi_x, H(V_{\omega, D})\, \phi_x\rangle] \\ &\quad = e^{-t\varepsilon_0} \exp[-t\langle\phi_x, V_\omega\phi_x\rangle] \exp[-tD\langle\phi_x, |X|^2\, \phi_x\rangle] \end{aligned} \tag{3.9}$$

because $\phi_x \in P_0 L^2(\mathbb{R}^2)$ belongs to the domain of $H(V_{\omega, D})$ for $\mathbb{P} \otimes$ *Lebesgue*-almost all pairs $(\omega, x) \in \Omega \times \mathbb{R}^2$. Since $X = (X_1, X_2)$ stands for the position operator, the expression

$$\langle\phi_x, |X|^2\, \phi_x\rangle = |x|^2 + 2\ell^2 \tag{3.10}$$

occurring in (3.9) is the quantum-mechanical mean-square distance of the particle from the origin in the vector state ϕ_x. Thanks to Fubini's theorem and the translation invariance of the random potential V_ω we thus get the following lower bound for the averaged "partition function"

$$\mathbb{E}[\operatorname{Tr} \mathrm{e}^{-tH(V_\omega,\, D)}] \geqslant \frac{1}{2\pi\ell^2}\, \mathrm{e}^{-t\varepsilon_0} \int_{\mathbb{R}^2} \mathrm{d}^2x \, \mathrm{e}^{-tD(|x|^2 + 2\ell^2)}\, \mathbb{E}[\mathrm{e}^{-t\langle \phi_x,\, V_\omega \phi_x\rangle}]$$

$$= \frac{1}{2\pi\ell^2} \frac{\pi}{tD}\, \mathrm{e}^{-t(\varepsilon_0 + 2D\ell^2)}\, \mathbb{E}[\mathrm{e}^{-t\langle \phi_0,\, V_\omega \phi_0\rangle}] \tag{3.11}$$

The desired inequality (3.5) now follows with the help of the approximation formula

$$\widetilde{N}(t) = \lim_{D \searrow 0} \frac{tD}{\pi}\, \mathrm{e}^{t\varepsilon_0}\, \mathbb{E}[\operatorname{Tr} \mathrm{e}^{-tH(V_\omega,\, D)}] \tag{3.12}$$

in (A.2) of ref. 2. ▮

For the quantum Lifshits tail Theorem 2.4 is concerned with the lower bound in (3.5) is still able to detect the leading low-energy asymptotics of N via the Tauberian Theorem A.1 in the Appendix. A complementing, much more involved upper bound is provided by the result of ref. 6. The key to profit by this result is the well-known fact that lessening the potential can only increase the integrated density of states, confer Theorem 5.24 in ref. 14.

Theorem 3.2. The integrated density of states obeys the inequality

$$N(E) \leqslant \mathbb{E}[\Theta(E - H(\hat{V}_\omega))(x, x)] \tag{3.13}$$

for any Poissonian potential $\hat{V}_\omega(x) := \sum_j \hat{U}(x - q_\omega(j))$ with $0 \leqslant \hat{U} \leqslant U$.

Proof. Using the fact that, according to (2.13), $N(E)$ and $\mathbb{E}[\Theta(E - H(\hat{V}_\omega))(x, x)]$ may be viewed as macroscopic limits the claimed inequality is a consequence of the min-max principle. ▮

Corollary 3.3. Under Assumptions (2.1) and (2.2) the low-energy asymptotics of the integrated density of states is bounded according to

$$\limsup_{E \searrow 0} \frac{\ln N(\varepsilon_0 + E)}{|\ln E|} \leqslant -2\pi\varrho\ell^2 \tag{3.14}$$

Proof. By smoothening the edge of the Heaviside unit-step function Θ in assumption (2.2), one can find some continuous function $\hat{U}\colon \mathbb{R}^2 \to [0, \infty[$ with

$$g_0 \Theta\left(\frac{r}{2} - |x-a|\right) \leqslant \hat{U}(x) \leqslant g_0 \Theta(r - |x-a|) \leqslant U(x) \tag{3.15}$$

Theorem 3.2 now allows to make contact with the situation considered in Theorem 1.1 of ref. 6 and use the known asymptotics, confer (1.9), of the right-hand side of (3.13). ∎

Before we proceed with the proofs of Theorems 2.3–2.5, it will be helpful to consider the following preparatory results.

3.2. Preparatory Results

To evaluate the long-time falloffs of the upper and lower bounds in (3.4) and (3.5), it is useful to know that one can compute the involved expectation values explicitly. This is most easily done with the help of the Laplace characteristic functional (2.8) by choosing a suitable J. The next lemma deals with the asymptotic behavior of certain integrals arising from the bounds in (3.4) and (3.5) after computing the Poissonian expectation values.

Lemma 3.4. Let W denote a positive integrable function on $\mathbb{R}^2$ which decays at infinity like

$$\lim_{|x| \to \infty} \frac{\ln W(x)}{|x|^\beta} = -\frac{1}{\mu^\beta} \tag{3.16}$$

for some $0 < \mu < \infty$ and $0 < \beta < \infty$. Then

$$\lim_{t \to \infty} (\ln t)^{-2/\beta} \int_{\mathbb{R}^2} d^2x \, (1 - e^{-tW(x)}) = \pi \mu^2 \tag{3.17}$$

Proof. Changing variables $x =: \mu (\ln t)^{1/\beta} \xi$ in the integral in (3.17), one gets

$$(\ln t)^{-2/\beta} \int_{\mathbb{R}^2} d^2x \, (1 - e^{-tW(x)}) = \mu^2 \int_{\mathbb{R}^2} d^2\xi \, (1 - e^{-tW(\mu (\ln t)^{1/\beta} \xi)}) \tag{3.18}$$

Assumption (3.16) assures that for every $0<\varepsilon<1$ there exists $T_\varepsilon>1$ such that

$$-1-\varepsilon \leqslant \frac{\ln W(\mu(\ln t)^{1/\beta}\,\xi)}{(\ln t)\,|\xi|^\beta} \leqslant -1+\varepsilon \tag{3.19}$$

for all $t>T_\varepsilon$, which implies

$$\begin{aligned} &1-\exp[-t^{1-(1+\varepsilon)\,|\xi|^\beta}] \\ &\quad \leqslant 1-\exp[-tW(\mu(\ln t)^{1/\beta}\,\xi)] \\ &\quad \leqslant \Theta(1-(1-\varepsilon)\,|\xi|^\beta)+t^{1-(1-\varepsilon)\,|\xi|^\beta}\Theta((1-\varepsilon)\,|\xi|^\beta-1) \end{aligned} \tag{3.20}$$

where we additionally used $1-\mathrm{e}^{-c}\leqslant c$ for the second inequality. The bounds (3.20) show that the integrand on the right-hand side of (3.18) converges pointwise inside and outside the unit disk

$$\lim_{t\to\infty}\left(1-\mathrm{e}^{-tW(\mu(\ln t)^{1/\beta}\,\xi)}\right)=\begin{cases}1 & \text{if } |\xi|<1\\ 0 & \text{if } |\xi|>1\end{cases} \tag{3.21}$$

The claimed result now follows from (3.18) in the limit $t\to\infty$ by interchanging limit and integration with the help of the dominated-convergence theorem and (3.21). Here the integrable bound in (3.20) assures that this theorem is indeed applicable. ▮

The previous Lemma 3.4 shows that the asymptotic behavior for large t of the lower bound in (3.5) depends on the long-distance decay of the convolution

$$\begin{aligned} (|\phi_0|^2 * U)(x) &:= \int_{\mathbb{R}^2} \mathrm{d}^2y\, |\phi_0(x-y)|^2\, U(y) \\ &= \frac{1}{2\pi\ell^2}\int_{\mathbb{R}^2} \mathrm{d}^2y\, \mathrm{e}^{-|x-y|^2/2\ell^2}\, U(y) \end{aligned} \tag{3.22}$$

In the next lemma we determine this decay for all three types of decay considered for U. Roughly speaking, the convolution is shown to decay like that convolving factor which has the slower decay.

Lemma 3.5. Let $j=1,2$ and $W_j\neq 0$ be positive integrable functions on $\mathbb{R}^2$ which decay at infinity like

$$\lim_{|x|\to\infty}\frac{\ln W_j(x)}{|x|^\beta}=-\frac{1}{\mu_j^\beta} \tag{3.23}$$

with some constants β, μ_j obeying either

(i) $0<\beta<\infty$, $0<\mu_1<\infty$ and $1/\mu_2=\infty$

or

(ii) $\beta=2$ and $0<\mu_j<\infty$.

Then

$$\lim_{|x|\to\infty} \frac{\ln(W_1 * W_2)(x)}{|x|^\beta} = -\frac{1}{\mu^\beta} \tag{3.24}$$

where $\mu := \mu_1$ in case (i), and $\mu := \sqrt{\mu_1^2+\mu_2^2}$ in case (ii).

Proof of Case (i) of Lemma 3.5. Our assumptions imply that for every $0<\varepsilon<1$ there is $R_\varepsilon>0$ such that

$$\exp\left[-(1+\varepsilon)\left(\frac{|x|}{\mu_1}\right)^\beta\right] \leqslant W_1(x) \leqslant \exp\left[-(1-\varepsilon)\left(\frac{|x|}{\mu_1}\right)^\beta\right] \tag{3.25}$$

and

$$W_2(x) \leqslant \exp\left[-\left(\frac{|x|}{\varepsilon\mu_1}\right)^\beta\right] \tag{3.26}$$

for all $|x|>R_\varepsilon$. To proof that the convolution $W_1 * W_2$ decays at least like W_1, we pick $r>0$ such that the intersection of the open disk $B(0,r)$, centered at the origin with radius r, and the support of W_2 contains a non-empty open set. By using (3.25) we estimate the convolution of W_1 and W_2 for every $|x|>R_\varepsilon+r$ as follows

$$\begin{aligned}(W_1 * W_2)(x) &\geqslant \inf_{|x-z|<r} W_1(z) \int_{|y|\leqslant r} d^2y\, W_2(y) \\ &\geqslant \exp\left[-(1+\varepsilon)\left(\frac{|x|+r}{\mu_1}\right)^\beta\right] \int_{|y|\leqslant r} d^2y\, W_2(y)\end{aligned} \tag{3.27}$$

Therefore the asymptotics of the convolution is bounded from below according to

$$\liminf_{|x|\to\infty} \frac{\ln(W_1 * W_2)(x)}{|x|^\beta} \geqslant -\frac{1+\varepsilon}{\mu_1^\beta} \tag{3.28}$$

To derive an upper bound we first choose $0<\varepsilon<1/2$. For $|x|>R_\varepsilon/\varepsilon$ we split the convolution integral into two integrals with domains of integration inside and outside the disk $B(0, \varepsilon\,|x|)$. With the help of (3.25) we may then estimate as follows

$$\begin{aligned}\int_{|y|\leqslant\varepsilon\,|x|} & d^2y\; W_1(x-y)\, W_2(y)\\ &\leqslant \int_{|y|\leqslant\varepsilon\,|x|} d^2y \exp\left[-(1-\varepsilon)\left(\frac{|x-y|}{\mu_1}\right)^{\beta}\right] W_2(y)\\ &\leqslant \|W_2\|_1 \exp\left[-(1-\varepsilon)\left(\frac{|x|\,(1-\varepsilon)}{\mu_1}\right)^{\beta}\right]\end{aligned} \tag{3.29}$$

where we used Hölder's inequality to bound the integral in the last step. The remaining term is treated similarly. Using (3.26) and Hölder's inequality we estimate

$$\begin{aligned}\int_{|y|>\varepsilon\,|x|} d^2y\; W_1(x-y)\, W_2(y) &\leqslant \int_{|y|>\varepsilon\,|x|} d^2y\; W_1(x-y) \exp\left[-\left(\frac{|y|}{\varepsilon\mu_1}\right)^{\beta}\right]\\ &\leqslant \|W_1\|_1 \exp\left[-\left(\frac{|x|}{\mu_1}\right)^{\beta}\right]\end{aligned} \tag{3.30}$$

Clearly, the first term dominates the asymptotics of the sum of the right-hand sides of (3.29) and (3.30), which yields

$$\limsup_{|x|\to\infty} \frac{\ln(W_1 * W_2)(x)}{|x|^{\beta}} \leqslant -\frac{(1-\varepsilon)^{\beta+1}}{\mu_1^{\beta}} \tag{3.31}$$

Since ε can be chosen arbitrarily small, case (i) of Lemma 3.5 is proved. ▮

Proof of Case (ii) of Lemma 3.5. Assumption (3.23) implies that for every $0<\varepsilon<1$ there exists $R_\varepsilon>0$ such that

$$\exp\left[-(1+\varepsilon)\frac{|x|^2}{\mu_j^2}\right] \leqslant W_j(x) \leqslant \exp\left[-(1-\varepsilon)\frac{|x|^2}{\mu_j^2}\right] \tag{3.32}$$

for all $|x|>R_\varepsilon$ and $j=1,2$. For a lower bound on $W_1 * W_2$ we use (3.32) to estimate the convolving factors on $\Xi(x,\varepsilon) := \mathbb{R}^2\backslash(B(0,R_\varepsilon)\cup B(x,R_\varepsilon))$, which yields

$$\begin{aligned}(W_1 * W_2)(x) &\geqslant \int_{\Xi(x,\varepsilon)} d^2y\; W_1(x-y)\, W_2(y)\\ &\geqslant \int_{\Xi(x,\varepsilon)} d^2y \exp\left\{-(1+\varepsilon)\left[\frac{|x-y|^2}{\mu_1^2}+\frac{|y|^2}{\mu_2^2}\right]\right\}\end{aligned} \tag{3.33}$$

By applying case (i) of the present lemma, the asymptotics of the last integral is seen to be unchanged if one replaces $\Xi(x, \varepsilon)$ by $\mathbb{R}^2$, since the remaining two terms stemming from integration over $B(0, R_\varepsilon)$ and $B(x, R_\varepsilon)$ decay like Gaussians with variances proportional to μ_1^2 and μ_2^2, respectively. From the explicit formula

$$\int_{\mathbb{R}^2} d^2y \exp\left\{-(1\pm\varepsilon)\left[\frac{|x-y|^2}{\mu_1^2}+\frac{|y|^2}{\mu_2^2}\right]\right\} = \frac{\pi}{(1\pm\varepsilon)}\frac{\mu_1^2\mu_2^2}{\mu_1^2+\mu_2^2}\exp\left[-(1\pm\varepsilon)\frac{|x|^2}{\mu_1^2+\mu_2^2}\right] \tag{3.34}$$

we then conclude

$$\liminf_{|x|\to\infty}\frac{\ln(W_1 * W_2)(x)}{|x|^2} \geqslant -\frac{1+\varepsilon}{\mu_1^2+\mu_2^2} \tag{3.35}$$

To obtain an upper bound we split the convolution integral into three terms by restricting the domain of integration to $B(0, R_\varepsilon)$, $B(x, R_\varepsilon)$ and $\Xi(x, \varepsilon)$ which are pairwise disjoint sets for all $|x| > 2R_\varepsilon$. The first two terms are estimated with the help of (3.32) for all $|x| > 2R_\varepsilon$ as follows

$$\int_{B(0, R_\varepsilon)} d^2y\, W_1(x-y)\, W_2(y) \leqslant \|W_2\|_1 \sup_{|x-y|\leqslant R_\varepsilon} W_1(y) \leqslant \|W_2\|_1 \exp\left[-(1-\varepsilon)\frac{(|x|-R_\varepsilon)^2}{\mu_1^2}\right] \tag{3.36}$$

$$\int_{B(x, R_\varepsilon)} d^2y\, W_1(x-y)\, W_2(y) \leqslant \|W_1\|_1 \sup_{|x-y|\leqslant R_\varepsilon} W_2(y) \leqslant \|W_1\|_1 \exp\left[-(1-\varepsilon)\frac{(|x|-R_\varepsilon)^2}{\mu_2^2}\right] \tag{3.37}$$

The integral over $\Xi(x, \varepsilon)$ dominates the sum of the three terms since it may be bounded using (3.34). Therefore we arrive at

$$\limsup_{|x|\to\infty}\frac{\ln(W_1 * W_2)(x)}{|x|^2} \leqslant -\frac{1-\varepsilon}{\mu_1^2+\mu_2^2} \tag{3.38}$$

which completes the proof since ε is arbitrary. ∎

3.3. Proofs of Theorems 2.3–2.5

Proof of Theorem 2.3. The claimed leading asymptotics of the integrated density of states is established with the help of the Tauberian Theorem A.1 and asymptotically coinciding lower and upper bounds on its shifted Laplace transform $\widetilde{N}$.

Lower Bound. With the help of (2.8) and inequality (3.5) we find

$$\liminf_{t\to\infty} \frac{\ln \widetilde{N}(t)}{(\ln t)^{2/\alpha}} \geqslant -\varrho \limsup_{t\to\infty} (\ln t)^{-2/\alpha} \int_{\mathbb{R}^2} d^2x \left(1-e^{-t(|\phi_0|^2 * U)(x)}\right) \tag{3.39}$$

Since U decays at infinity slower than any Gaussian, Lemma 3.5(i) implies

$$\lim_{|x|\to\infty} \frac{\ln(|\phi_0|^2 * U)(x)}{|x|^\alpha} = -\frac{1}{\lambda^\alpha} \tag{3.40}$$

which by employing Lemma 3.4 gives

$$\lim_{t\to\infty} (\ln t)^{-2/\alpha} \int_{\mathbb{R}^2} d^2x \left(1-e^{-t(|\phi_0|^2 * U)(x)}\right) = \pi\lambda^2 \tag{3.41}$$

Upper Bound. By means of (2.8) inequality (3.4) shows

$$\begin{aligned}\limsup_{t\to\infty} \frac{\ln \widetilde{N}(t)}{(\ln t)^{2/\alpha}} &\leqslant -\varrho \liminf_{t\to\infty} (\ln t)^{-2/\alpha} \int_{\mathbb{R}^2} d^2x \left(1-e^{-tU(x)}\right)\\ &= -\pi\varrho\lambda^2\end{aligned} \tag{3.42}$$

where we used again Lemma 3.4 in the last step. ∎

Proof of Theorem 2.4. We construct asymptotically coinciding lower and upper bounds on the integrated density of states N.

Lower Bound. Employing the inequality (3.5) together with (2.8) we find

$$\liminf_{t\to\infty} \frac{\ln \widetilde{N}(t)}{\ln t} \geqslant -\varrho \limsup_{t\to\infty} (\ln t)^{-1} \int_{\mathbb{R}^2} d^2x \left(1-e^{-t(|\phi_0|^2 * U)(x)}\right) \tag{3.43}$$

Since U decays faster than any Gaussian, Lemma 3.5(i) implies

$$\lim_{|x|\to\infty} \frac{\ln(|\phi_0|^2 * U)(x)}{|x|^2} = -\frac{1}{2\ell^2} \tag{3.44}$$

which by Lemma 3.4 gives

$$\lim_{t\to\infty} (\ln t)^{-1} \int_{\mathbb{R}^2} d^2x \left(1 - e^{-t(|\phi_0|^2 * U)(x)}\right) = 2\pi\ell^2 \tag{3.45}$$

To extract a lower bound on the leading asymptotics of the integrated density of states we use the Tauberian theorem (A.1) to obtain

$$\liminf_{E\searrow 0} \frac{\ln N(\varepsilon_0 + E)}{|\ln E|} = -2\pi\varrho\ell^2 \tag{3.46}$$

Upper Bound. It is provided by Corollary 3.3. ∎

Proof of Theorem 2.5. The claimed bounds on the leading asymptotics of the integrated density of states follow again by using suitable bounds on the shifted Laplace transform and applying the Tauberian Theorem A.1.

Lower Bound. The inequality (3.5) together with (2.8) and Lemma 3.4 yields

$$\liminf_{t\to\infty} \frac{\ln \widetilde{N}(t)}{\ln t} \geqslant -\varrho \limsup_{t\to\infty} (\ln t)^{-1} \int_{\mathbb{R}^2} d^2x \left(1 - e^{-t(|\phi_0|^2 * U)(x)}\right)$$

$$= -\pi\varrho(\lambda^2 + 2\ell^2) \tag{3.47}$$

Here we used the fact that U decays like a Gaussian so that

$$\lim_{|x|\to\infty} \frac{\ln(|\phi_0|^2 * U)(x)}{|x|^2} = -\frac{1}{\lambda^2 + 2\ell^2} \tag{3.48}$$

by Lemma 3.5(ii).

Upper Bound. Since Gaussian decay (2.5) implies assumption (2.2), Corollary 3.3 provides the claimed upper bound for $2\ell^2 \geqslant \lambda^2$. On the other hand, we may use inequality (3.4) together with Lemma 3.4 to achieve

$$\limsup_{t\to\infty} \frac{\ln \widetilde{N}(t)}{\ln t} \leqslant -\varrho \liminf_{t\to\infty} (\ln t)^{-1} \int_{\mathbb{R}^2} d^2x \left(1 - e^{-tU(x)}\right)$$

$$= -\pi\varrho\lambda^2 \quad ∎ \tag{3.49}$$

APPENDIX: A TAUBERIAN THEOREM

Theorem A.1. Let N be a distribution function on the real line $\mathbb{R}$. Assume there is a constant $\eta \in \mathbb{R}$ such that $N(E)=0$ for all $E \leqslant \eta$. Moreover, define the shifted Laplace transform of N by

$$\widetilde{N}(t) := \mathrm{e}^{t\eta} \int_{\eta}^{\infty} \mathrm{d}N(E)\, \mathrm{e}^{-tE}, \qquad t > 0 \tag{A.1}$$

and suppose that $\widetilde{N}(\tau) < \infty$ for some $\tau > 0$. Finally, let $\gamma \geqslant 1$ and $C > 0$. Then

$$\lim_{t \to \infty} \frac{\ln \widetilde{N}(t)}{(\ln t)^{\gamma}} = -C \tag{A.2}$$

if and only if

$$\lim_{E \searrow 0} \frac{\ln N(\eta + E)}{|\ln E|^{\gamma}} = -C \tag{A.3}$$

Proof. Without loss of generality we may assume $\eta = 0$. It is enough to show that Eq. (A.2) or Eq. (A.3) implies that $-C$ is an upper and lower bound on the lim sup and lim inf of the fraction on the left-hand side of the respective other equation. This can easily be achieved with the help of the following two lemmata. ∎

Lemma A.2. Let N be a distribution function with support in the positive half-line and $\widetilde{N}$ its (unshifted) Laplace transform. Then for all $E > 0$

$$N(E) \leqslant \mathrm{e}\, \widetilde{N}\left(\frac{1}{E}\right) \tag{A.4}$$

Proof. With the help of $\Theta(E') \leqslant \exp(tE')$ which is valid for all $E' \in \mathbb{R}$ and any $t \geqslant 0$ one obtains

$$N(E) = \int_{0}^{\infty} \mathrm{d}N(E')\, \Theta(E - E') \leqslant \mathrm{e}^{tE} \int_{0}^{\infty} \mathrm{d}N(E')\, \mathrm{e}^{-tE'} = \mathrm{e}^{tE}\, \widetilde{N}(t) \tag{A.5}$$

The choice $t = 1/E$ yields the lemma. ∎

For a lower bound on the distribution function N we need to evaluate the Laplace transform at a point slightly away from $1/E$, so we define for any $E>0$

$$t_E := \frac{1}{E}\,|\ln E|^{\gamma+1} \tag{A.6}$$

Lemma A.3. In the situation of Lemma A.2 suppose that either (A.2) or (A.3) (with η replaced by zero) holds. Furthermore, if there is a $\tau>0$ such that $\tilde{N}(\tau)<\infty$, then the inequality

$$N(E) \geqslant \tilde{N}(t_E) - 3\,\mathrm{e}^{-|\ln E|^{\gamma+1}} \tag{A.7}$$

holds for sufficiently large $1/E$.

Proof. By splitting the domain of integration of the Laplace transformation and employing the fact that 1 is an upper bound on the exponential of a negative argument we can write

$$\tilde{N}(t_E) \leqslant N(E) + \int_E^\infty \mathrm{d}N(E')\,\mathrm{e}^{-t_E E'} \tag{A.8}$$

After re-ordering the terms we are finished if we can show that the remaining integral does not exceed $3\exp(-|\ln E|^{\gamma+1})$ for large $1/E$. Therefore we note that either (A.3) directly or (A.2) via Lemma A.2 assures that there exists $E_0>0$ such that

$$\frac{\ln N(E)}{|\ln E|^\gamma} \leqslant -\frac{C}{2} \tag{A.9}$$

for every $0<E\leqslant E_0$. Now we pick an arbitrary but fixed $E_1>0$ such that

$$E_1 < E_0, \qquad E_1 < \mathrm{e}^{-\sqrt{C\gamma/2}} \qquad \text{and} \qquad t_{E_1} \geqslant 2\tau \tag{A.10}$$

In the following we will assume $0<E<E_1$. After these preparations we are ready to estimate the upper part of the integral in (A.8)

$$\begin{aligned}\int_{E_1}^\infty \mathrm{d}N(E')\,\mathrm{e}^{-tE'} &\leqslant \mathrm{e}^{-tE_1/2}\int_{E_1}^\infty \mathrm{d}N(E')\,\mathrm{e}^{-tE'/2}\\ &\leqslant \mathrm{e}^{-tE_1/2}\int_0^\infty \mathrm{d}N(E')\,\mathrm{e}^{-tE'/2} \leqslant \mathrm{e}^{-tE_1/2}\,\tilde{N}(t/2)\end{aligned} \tag{A.11}$$

Substituting t_E for t and using $t_E/2 \geqslant \tau$ and the fact that $\tilde{N}$ is monotone decreasing (due to monotonicity of N) one arrives at

$$\int_{E_1}^{\infty} \mathrm{d}N(E')\, \mathrm{e}^{-t_E E'} \leqslant \mathrm{e}^{-t_E E_1/2}\, \tilde{N}(\tau) \leqslant \mathrm{e}^{-1/E} \leqslant \mathrm{e}^{-|\ln E|^{\gamma+1}} \tag{A.12}$$

for $1/E$ sufficiently large as t_E grows faster than $1/E$ with decreasing $E<1$. Next we have to consider the lower part of the integral in (A.8)

$$\int_{E}^{E_1} \mathrm{d}N(E')\, \mathrm{e}^{-t_E E'} = t_E \int_{E}^{E_1} \mathrm{d}E'\, N(E')\, \mathrm{e}^{-t_E E'} + [N(E')\, \mathrm{e}^{-t_E E'}]_{E'=E}^{E_1} \tag{A.13}$$

where the equality comes from an integration by parts. The last term can be bounded according to

$$[N(E')\, \mathrm{e}^{-t_E E'}]_{E'=E}^{E_1} \leqslant N(E_1)\, \mathrm{e}^{-t_E E_1} \leqslant \mathrm{e}^{-1/E} \leqslant \mathrm{e}^{-|\ln E|^{\gamma+1}} \tag{A.14}$$

for sufficiently large $1/E$. In the first term on the right-hand side of (A.13) the integration extends only over E' smaller than E_1 and hence smaller than E_0. Thus (A.9) can be applied to obtain $\exp(-t_E f_{t_E}(E'))$ as an upper bound on the integrand. Here the function f_t is defined by

$$f_t(E') := E' + \frac{C}{2t} |\ln E'|^{\gamma} \tag{A.15}$$

Note that f_{t_E} is monotone increasing in the closed interval $[E, E_1]$ which can be seen by taking the derivative and using $\gamma \geqslant 1$ and $|\ln E'|^{-2} \leqslant |\ln E_1|^{-2} < 2/C\gamma$, see (A.10). Therefore we arrive at

$$\int_{E}^{E_1} \mathrm{d}E'\, N(E')\, \mathrm{e}^{-t_E E'} \leqslant (E_1 - E)\, \mathrm{e}^{-t_E f_{t_E}(E)} \tag{A.16}$$

By inserting the definitions of t_E and f_t one can thus check that the first term on the right-hand side of (A.13) is also bounded from above by $\exp(-|\ln E|^{\gamma+1})$ for $1/E$ large enough. ▮

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NOTE ADDED IN PROOF

In a recent work,[9] which complements the present one, we have determined rather explicitly the magnetic Lifshits tails for all impurity potentials U with so-called *regular* sub-Gaussian long-distance decay. The resulting formula covers a great variety of classical Lifshits tails and in particular (1.7) and (2.16) for algebraic and stretched-Gaussian decay. Roughly speaking, a sub-Gaussian decay is regular if it has no severe oscillations. Unfortunately, if $U \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{L}^2(\mathbb{R}^2)$ has sub-Gaussian but not regular decay, we do not know how to determine the Lifshits tail. In this case, we cannot even rule out that it exhibits quantum effects.

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